Part 2: Question 1 (20 points)

Once per day, a tour boat operator brings I tourists on a boat to an island with the ruins of an old monastery. Upon arriving at the island, the tourists get off the boat and explore the island. Once all of the tourists have finished exploring the island they return to the boat and the operator brings them back to the mainland. Due to legal reasons, the tour boat operator must wait for all tourists to return to the boat and transport all I tourists back to the mainland together, all at the same time.

Each tourist initially enjoys spending time on the island, but eventually after exploring all of it they become bored and want to return to the mainland. The tourists have heterogeneous preferences for time spent on the island. The derived utility for tourist i with wealth w_i from spending $t \geq 0$ units of time on the island is given by:

$$\phi_i(t, w_i) = \gamma_i t - \frac{1}{2}t^2 + w_i$$

where $0 < \gamma_1 < \gamma_2 < \cdots < \gamma_I$. Tourist i does not have complete control over t. The boat only returns to the mainland once all I tourists have returned to the boat. Therefore if each tourist i chooses to spend t_i units of time on the island, the boat will only leave after $t = \max\{t_1, t_2, \ldots, t_I\}$ units of time.

(i) [**5 points**] Show that if each tourist individually decides how long to spend on the island (not taking into account their impact on other tourists), the boat will spend $t^\star = \gamma_I$ units of time on the island.

(ii) [**5 points**] Show that a social planner who values each consumer's utility equally will choose that the boat spends $t^\circ = \frac{1}{I}\sum_{i=1}^{I} \gamma_i$ units of time on the island.

(iii) [**3 points**] Give an intuitive reason for why $t^\star \neq t^\circ$.

(iv) [**7 points**] In order to correct for this welfare loss, the tour boat operator decides to charge each tourist a personalized price p_i for each unit of time spent on the island. The prices are chosen such that all tourists choose to return to the boat after t° units of time. The price p_i may be negative for some consumers, in which case the tour boat operator pays the consumer for each unit of time spent on the island. Find a set of prices p_i and lump-sum transfers T_i for each consumer such that each consumer receives the same utility as in the planner's problem in part (ii).

Part 2: Question 2 (20 Points)

Consider a pure exchange economy with $I > 1$ consumers and $L > 1$ goods. Each consumer i's preferences are complete, transitive, continuous, strictly convex and strongly monotone. Each consumer's consumption set is $X_i = \mathbb{R}^L_+$. There is a single firm whose only production technology is free disposal: $\mathcal{Y}_1 = \left\{\boldsymbol{y}_1 \in \mathbb{R}^L : \boldsymbol{y}_1 \leq \boldsymbol{0}\right\}$. The aggregate endowment vector satisfies $\bar{\boldsymbol{\omega}} \gg \boldsymbol{0}$ (strictly positive in all goods).

In this economy there is an allocation $(\boldsymbol{x}_1^\star, \ldots, \boldsymbol{x}_I^\star, \boldsymbol{y}_1^\star)$ with $\boldsymbol{x}_i^\star \gg \boldsymbol{0}$ for all i and $\boldsymbol{y}_1^\star = \boldsymbol{0}$ that is Pareto optimal.

In this question, we want to prove that if $\boldsymbol{\omega}_i = \boldsymbol{x}_i^\star$ for all i, then $(\boldsymbol{x}_1^\star, \ldots, \boldsymbol{x}_I^\star, \boldsymbol{y}_1^\star)$ with $\boldsymbol{y}_1^\star = \boldsymbol{0}$ is the unique Walrasian equilibrium in this economy.

To prove this, we split it into a number of steps.

(i) [**3 points**] Based on the assumptions above, we are guaranteed that an equilibrium exists in this economy (using the existence proof from class). Call the allocation in this equilibrium $(\boldsymbol{x}_1', \ldots, \boldsymbol{x}_I', \boldsymbol{y}_1')$ and the price vector $\boldsymbol{p}'$. Show that $\boldsymbol{p}' \gg \boldsymbol{0}$ (i.e. argue that we cannot have any $p_\ell = 0$).

(ii) [**3 points**] Using (i), show that $\boldsymbol{y}_1' = \boldsymbol{0}$.

(iii) [**3 points**] Show that $\boldsymbol{x}_i' \sim_i \boldsymbol{x}_i^\star$ for all i. *Hint:* Use the fact that $\boldsymbol{\omega}_i = \boldsymbol{x}_i^\star$ for all i and that $(\boldsymbol{x}_1^\star, \ldots, \boldsymbol{x}_I^\star, \boldsymbol{y}_1^\star)$ is Pareto optimal.

(iv) [**3 points**] Use (ii) and (iii) above to show that $(\boldsymbol{x}_1^\star, \ldots, \boldsymbol{x}_I^\star, \boldsymbol{y}_1^\star)$ with price vector $\boldsymbol{p}'$ is a Walrasian equilibrium.

(v) [**8 points**] Using the fact that $(\boldsymbol{x}_1^\star, \ldots, \boldsymbol{x}_I^\star, \boldsymbol{y}_1^\star)$ is a Walrasian equilibrium allocation from part (iv), prove that this equilibrium allocation is unique. *Hint:* Use proof by contradiction. Suppose there is another allocation that is also an equilibrium, and make use of the strict convexity of preferences.